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Solving Systems of Equations

1. Solve this system, and write its solution set *both* in parametric form
and in parametric vector form.

$$\left\{ \begin{array}{l} x_1 + x_3 + 2x_4 = 3 \\ x_2 + x_3 + 3x_4 = 2 \\ 2x_1 - x_2 + x_3 + x_4 = 4 \end{array} \right.$$

reduce

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 3 \\ 0 & 1 & 1 & 3 & 2 \\ 2 & -1 & 1 & 1 & 4 \end{array} \right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 3 \\ 0 & 1 & 1 & 3 & 2 \\ 0 & -1 & -1 & -3 & -2 \end{array} \right] R_3 - 2R_1$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 3 \\ 0 & 1 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] R_3 + R_2$$

 \Leftrightarrow

$$\left\{ \begin{array}{l} x_1 + x_3 + 2x_4 = 3 \\ x_2 + x_3 + 3x_4 = 2 \\ x_3 \text{ free} \\ x_4 \text{ free} \end{array} \right.$$

$$\Leftrightarrow \left\{ \begin{array}{l} x_1 = 3 - x_3 - 2x_4 \\ x_2 = 2 - x_3 - 3x_4 \\ x_3 \text{ free} \\ x_4 \text{ free} \end{array} \right.$$

Solutions

$$\vec{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 - x_3 - 2x_4 \\ 2 - x_3 - 3x_4 \\ 0 + x_3 + 0 \\ 0 + 0 + x_4 \end{bmatrix}$$

$$\vec{X} = \begin{bmatrix} 3 \\ 2 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

for all $x_3, x_4 \in \mathbb{R}$

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1. Fill in the blanks, to complete the statement of Theorem 2:

Theorem 2: The reduced echelon form of a linear system has three possible cases

- (a) The system has zero solutions if it contains a row $[0 \dots 0 | \square]$
exactly one pivot in every coeff. column.
- (b) The system has one solutions if it is consistent and has a pivot in every coeff. column.
- (c) The system has ∞ -many solutions if if some coeff. column does NOT contain a pivot

2. For each of the cases above, write down an augmented matrix with the corresponding number of solutions.

(a) zero
$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 4 \end{array} \right]$$

requires
 $0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 = 4$
but $0 \neq 4$

(b) unique
$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 9 \end{array} \right]$$

(c) ∞ -many
$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

↑
(free variable)
in column 2

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4. Give an example of the augmented matrix of a system of 2 equations in 3 variables ...
 (Hint: you can make the systems as simple as you like.)

(a) with no solutions

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{array} \right] \quad \xrightarrow{\text{has no solution}} \quad \begin{cases} x_1 + 2x_3 = 3 \\ 0 = 1 \end{cases}$$

(b) with infinitely many solutions.

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & 4 & 2 \end{array} \right] \quad \xrightarrow{\text{no pivot}} \quad \begin{cases} x_1 + 2x_3 = 3 \\ x_2 + 4x_3 = 2 \\ x_3 \text{ free} \end{cases}$$

5. Use Theorem 2 to prove that you cannot write a system of 2 equations in 3 variables that has a unique solution. (Hint: your argument must consider all relevant reduced-echelon form matrices.)

By theorem 2,

have unique solution \Leftrightarrow have pivot in each coefficient columnBut consider $\left[\begin{array}{ccc|c} * & * & * & * \\ * & * & * & * \end{array} \right]$

you cannot fit 3 pivots in 2 rows

Therefore you cannot have a unique solution

6. Where possible, give an example of the augmented matrix of a system of 3 equations in 3 variables with 0 solutions, 1 solution, and infinitely many solutions.

If one of these cases are impossible, use Theorem 2 to explain why.

No solutions:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -3 \end{array} \right]$$

unique solution

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 1 \\ 0 & 6 & 2 & 3 \\ 0 & 0 & 1 & 0 \end{array} \right]$$

Pivot in each coeff column

>> many solutions

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

no pivot

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4. (No Computation) For each of the following matrices, determine if its augmented matrix is in echelon form, reduced echelon form, or neither. If it is in echelon form, indicate which *columns* contain pivots.

(a)

$$\left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right]$$

Not in either - pivots are not stepping
in order

(b)

$$\left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right]$$

reduced echelon form
pivot in every column

(c)

$$\left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{array} \right]$$

echelon form but NOT reduced echelon form
pivots in columns 1, 2, & 3.

(d)

$$\left[\begin{array}{cc|cc} 1 & 0 & 4 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right]$$

reduced echelon form.
pivots in columns 1, 2, 4.

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1. (a) Write the following augmented matrix as a vector equation, a matrix equation, and a system of linear equations.

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & -3 \\ 2 & 3 & 1 & 1 \end{array} \right]$$

$$x_1 \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \cdot \begin{bmatrix} 2 \\ 3 \end{bmatrix} + x_3 \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix} \quad \left[\begin{array}{ccc|c} 1 & 2 & -1 & -3 \\ 2 & 3 & 1 & 1 \end{array} \right] \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

3 cases $\begin{cases} 2pt \text{ for all 3} \\ 1pt \text{ for 2} \\ 0.5 pt \text{ for 1.} \end{cases}$

$$\begin{cases} x_1 + 2x_2 - x_3 = -3 \\ 2x_1 + 3x_2 + x_3 = 1 \end{cases}$$

- (b) Is the system of equations consistent?

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & -3 \\ 0 & -1 & 3 & 7 \end{array} \right] \xrightarrow{r_2 - 2r_1} \begin{matrix} \text{echelon form} \\ \text{doesn't contain} \\ [0 \dots 0 | \blacksquare] \\ \Rightarrow \text{consistent} \end{matrix}$$

1pt

1pt

- (c) Solve the above system. (Find the solution set).

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 5 & 11 \\ 0 & 1 & 3 & 7 \end{array} \right] \xrightarrow{r_1 + 2r_2} \sim \left[\begin{array}{ccc|c} 1 & 0 & 5 & 11 \\ 0 & 1 & -3 & -7 \end{array} \right] \quad \begin{matrix} 2pt \text{ for final matrix} \\ \text{2pt for parametric form} \end{matrix}$$

$$\Leftrightarrow \begin{cases} x_1 + 5x_3 = 11 \\ x_2 - 3x_3 = -7 \\ x_3 \text{ free} \end{cases} \quad \Leftrightarrow \begin{cases} x_1 = 11 - 5x_3 \\ x_2 = -7 + 3x_3 \\ x_3 \text{ free} \end{cases}$$

- (d) If the system is consistent, write down a particular solution.

Verify that this is a solution *two different ways* by plugging it into both
i. the vector equation, and ii. the system of equation.

Pick $x_3 = 0$

$$\Rightarrow \vec{x} = \begin{bmatrix} 11 \\ -7 \\ 0 \end{bmatrix}$$

$$11 \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} + (-7) \begin{bmatrix} 2 \\ 3 \end{bmatrix} + 0 \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 11 \\ 22 \end{bmatrix} + \begin{bmatrix} -14 \\ -21 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix} \quad \checkmark$$

$$\begin{cases} 11 + 2 \cdot (-7) - 0 = 11 - 14 = -3 \checkmark \\ 2 \cdot 11 + 3 \cdot (-7) + 0 = 22 - 21 = 1 \checkmark \end{cases}$$

1pt each

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2. (a) Write the following augmented matrix as a vector equation, a matrix equation, and a system of linear equations.

$$x_1 \cdot \begin{bmatrix} 0 \\ 5 \\ 3 \end{bmatrix} + x_2 \cdot \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ 6 \end{bmatrix} \quad \left| \quad \begin{bmatrix} 0 & 4 & | & 6 \\ 5 & 2 & | & 10 \\ 3 & 3 & | & 6 \end{bmatrix} \right.$$

$$\left| \quad \begin{bmatrix} 0 & 4 \\ 5 & 2 \\ 3 & 3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ 6 \end{bmatrix} \right. \quad \left| \quad \begin{cases} 4x_2 = 6 \\ 5x_1 + 2x_2 = 10 \\ 3x_1 + 3x_2 = 6 \end{cases} \right.$$

- (b) Is the system of equations consistent?

$$\sim \left[\begin{array}{cc|c} 3 & 3 & 6 \\ 5 & 2 & 10 \\ 0 & 4 & 6 \end{array} \right] \sim \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 5 & 2 & 10 \\ 0 & 2 & 3 \end{array} \right] \xrightarrow{\frac{1}{2}r_3} \sim \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & -3 & 0 \\ 0 & 2 & 3 \end{array} \right] \xrightarrow{r_2 - 5r_1} \sim \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 2 & 3 \end{array} \right] \xrightarrow{r_3 - 2r_1}$$

INCONSISTENT

- (c) Solve the above system. (Find the solution set).

requires $0=3$



no solutions

(empty set)

- (d) If the system is consistent, write down a particular solution.

Verify that this is a solution *three different ways* by plugging it into *i.* the vector equation, *ii.* matrix equation, and *iii.* system of equation.

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3. (a) Write the following augmented matrix as a vector equation, a matrix equation, and a system of linear equations.

$$\left[\begin{array}{ccc|c} 3 & -3 & 3 & 3 \\ -1 & 2 & 1 & 3 \\ 2 & 8 & 2 & 2 \end{array} \right]$$

$$x_1 \cdot \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} + x_2 \cdot \begin{bmatrix} -3 \\ 2 \\ 8 \end{bmatrix} + x_3 \cdot \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -3 & 3 \\ -1 & 2 & 1 \\ 2 & 8 & 2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix}$$

$$\begin{cases} 3x_1 - 3x_2 + 3x_3 = 3 \\ -x_1 + 2x_2 + x_3 = 3 \\ 2x_1 + 8x_2 + 2x_3 = 2 \end{cases}$$

- (b) Is the system of equations consistent?

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ -1 & 2 & 1 & 3 \\ 1 & 4 & 1 & 1 \end{array} \right] \begin{matrix} r_3 + r_1 \\ r_2 + r_1 \\ r_3 - r_1 \end{matrix}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 1 & 2 & 4 \\ 0 & 5 & 0 & 0 \end{array} \right] \begin{matrix} r_2 + r_1 \\ r_3 - r_1 \end{matrix}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & 0 & 0 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 4 \end{array} \right]$$

Step form

consistent
because
only 0's
 $[0 \ 0 \ 0 \ | \ 1]$

- (c) Solve the above system. (Find the solution set).

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{array} \right] \begin{matrix} r_1 + r_2 \\ r_2 - r_3 \\ r_1 - r_3 \end{matrix}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right]$$

$$\begin{cases} x_1 = -1 \\ x_2 = 0 \\ x_3 = 2 \end{cases}$$

- (d) If the system is consistent, write down a particular solution.

Verify that this is a solution *three different ways* by plugging it into *i.* the vector equation, *ii.* matrix equation, and *iii.* system of equation.

vector equation

$$-1 \cdot \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} + 0 \cdot \begin{bmatrix} -3 \\ 2 \\ 8 \end{bmatrix} + 2 \cdot \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ -2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 6 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix} \checkmark$$

system of eqns

$$\begin{cases} 3 \cdot (-1) + -3 \cdot 0 + 3 \cdot 2 = -3 + 6 = 3 \\ -1 \cdot (-1) + 2 \cdot 0 + 1 \cdot 2 = 1 + 2 = 3 \\ 2 \cdot (-1) + 8 \cdot 0 + 2 \cdot 2 = -2 + 4 = 2 \end{cases} \checkmark$$

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Vectors and Vector Equations

1. (No Computation) Write down the formal definition of $\text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

$$\text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \left\{ \begin{array}{l} \text{the set of all} \\ \vec{b} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 : \text{for } c_1, c_2, c_3 \in \mathbb{R} \end{array} \right\}$$

2. (No Computation) What is the graphical meaning of $\text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$?

it is the set of all vectors that
can be hit by scaling & adding $\vec{v}_1, \vec{v}_2, \text{ and } \vec{v}_3$

3. Describe the Span of the following sets of vectors:

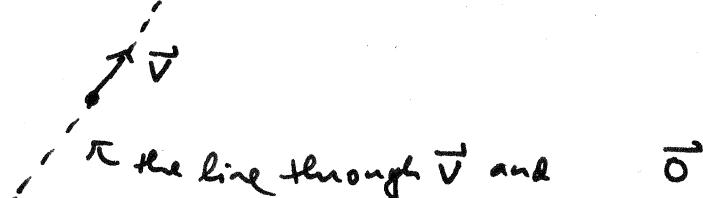
- (a) $\text{Span}\{\vec{v}\}$ where $\vec{v} = \vec{0}$

$$\{t \cdot \vec{0} : t \text{ is in } \mathbb{R}\} = \{\vec{0}\}$$

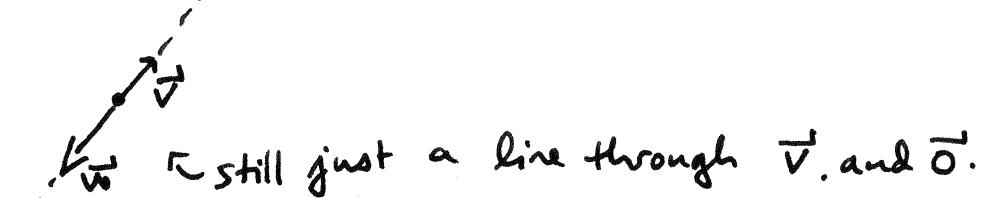
\hookrightarrow a single point

set containing ONLY
the zero vectn.

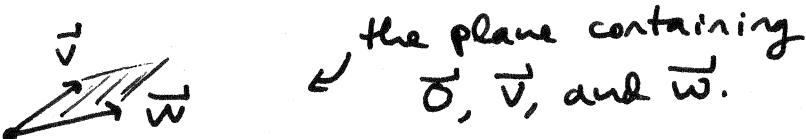
- (b) $\text{Span}\{\vec{v}\}$ where $\vec{v} \neq \vec{0}$



- (c) $\text{Span}\{\vec{v}, \vec{w}\}$ where $\vec{v}, \vec{w} \neq \vec{0}$ and $\vec{w} = k\vec{v}$ for some $k \in \mathbb{R}$.



- (d) $\text{Span}\{\vec{v}, \vec{w}\}$ where $\vec{v}, \vec{w} \neq \vec{0}$ and $\vec{w} \neq k\vec{v}$ for every $k \in \mathbb{R}$.



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4. Is the vector $\vec{b} = \begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix}$ in the span of the vectors $\vec{a}_1 = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$, $\vec{a}_2 = \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix}$, $\vec{a}_3 = \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix}$?

i.e. is $x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3 = \vec{b}$ consistent?

reduce

$$\left[\begin{array}{ccc|c} 1 & -1 & -2 & 2 \\ 3 & -3 & -3 & 3 \\ -2 & 2 & 1 & 6 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & -1 & -2 & 2 \\ 1 & -1 & -1 & 1 \\ -2 & 2 & 1 & 6 \end{array} \right] \xrightarrow{r_3 \leftarrow r_2} \\ \sim \left[\begin{array}{ccc|c} 1 & -1 & -2 & 2 \\ 0 & 0 & 1 & -1 \end{array} \right] \xrightarrow{r_2 \leftarrow r_1} \\ \left[\begin{array}{ccc|c} 1 & -1 & -2 & 2 \\ 0 & 0 & -3 & 10 \end{array} \right] \xrightarrow{r_3 \leftarrow r_3 + 2r_1} \\ \sim \left[\begin{array}{ccc|c} 1 & -1 & 2 & 2 \\ 0 & 0 & 1 & -1 \end{array} \right] \xrightarrow{r_3 \leftarrow r_3 + 3r_2} \quad \text{No solution, by theorem 2}$$

So vector is NOT in $\text{span}\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$

5. Find h so that the vector the vector $\vec{b} = \begin{bmatrix} 2 \\ h \\ 3 \end{bmatrix}$ in the span of the vectors

$$\vec{a}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \vec{a}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \vec{a}_3 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}.$$

want

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 1 & -1 & 0 & h \\ -1 & 1 & 0 & 3 \end{array} \right] \text{ to be consistent}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & -2 & -2 & h-2 \end{array} \right] \xrightarrow{r_2 \leftarrow r_2} \\ \left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & 2 & 2 & 5 \end{array} \right] \xrightarrow{r_3 \leftarrow r_3 + r_1}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & -2 & -2 & h-2 \\ 0 & 0 & 0 & h+3 \end{array} \right] \xrightarrow{r_3 \leftarrow r_3 + r_2}$$

this is consistent $\Leftrightarrow h+3=0$

$$\Leftrightarrow h = -3$$

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6. Write the Solution Set of the equation

$$x_1 \begin{bmatrix} 2 \\ -2 \\ -4 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} + x_3 \begin{bmatrix} 4 \\ -1 \\ -6 \end{bmatrix} = \begin{bmatrix} 4 \\ \boxed{} \\ -4 \end{bmatrix}$$

in parametric vector form.

reduce

$$\left[\begin{array}{ccc|c} 2 & 2 & 4 & 4 \\ -2 & 1 & -1 & \boxed{} \\ -4 & -2 & -6 & -4 \end{array} \right]$$

$$\sim \left[\begin{array}{ccc|c} 2 & 2 & 4 & 4 \\ 0 & 3 & 3 & 6 \\ 0 & 2 & 2 & 4 \end{array} \right] \begin{matrix} r_2+r_1 \\ r_3+2r_1 \end{matrix}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 2 \end{array} \right] \begin{matrix} r_2-r_1 \\ r_3-r_1 \\ r_3-r_2 \end{matrix}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array} \right] \begin{matrix} r_1-r_2 \\ r_3-r_2 \end{matrix}$$

 \Leftrightarrow

$$\begin{cases} x_1 + x_3 = 0 \\ x_2 + x_3 = 2 \\ x_3 \text{ free} \end{cases}$$

 \Leftrightarrow

$$\begin{cases} x_1 = -x_3 \\ x_2 = 2 - x_3 \\ x_3 \text{ free} \end{cases}$$

Solutions are

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & -x_3 \\ 2 & -x_3 \\ 0 & x_3 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

for all x_3 in \mathbb{R}

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2. Do the vectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 2 \\ 0 \\ 8 \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$ span \mathbb{R}^3 ?

i.e. is $[\vec{v}_1 \vec{v}_2 | \vec{v}_3 | \vec{b}]$ consistent for all \vec{b}

i.e. is $\left[\begin{array}{ccc|c} 1 & 2 & 1 & b_1 \\ 1 & 0 & 3 & b_2 \\ 3 & 8 & 1 & b_3 \end{array} \right]$ consistent for all b_1, b_2, b_3 ?

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & b_1 \\ 0 & -2 & 2 & b_2 - b_1 \\ 0 & 2 & -2 & b_3 - 3b_1 \end{array} \right] \xrightarrow{r_2 - r_1, r_3 - 3r_1}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & b_1 \\ 0 & -2 & 2 & b_2 - b_1 \\ 0 & 0 & 0 & b_3 - b_1 + b_3 - 3b_1 \end{array} \right] \xrightarrow{r_3 + r_2}$$

is only consistent when
 $b_2 + b_3 - 4b_1 = 0$

But this does not always hold

so $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$
do NOT span \mathbb{R}^3

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3. Find all possible h so that the vectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 3 \\ h \end{bmatrix}$ span \mathbb{R}^2 .

i.e. find h so that

$$\left[\begin{array}{cc|c} 1 & 3 & b_1 \\ 2 & h & b_2 \end{array} \right] \text{ is consistent for all } b_1, b_2$$

$$\sim \left[\begin{array}{cc|c} 1 & 3 & b_1 \\ 0 & h-6 & b_2 - 2b_1 \end{array} \right] \xrightarrow{r_2 - 2r_1}$$

NOTE:

you can make $b_2 - 2b_1 \neq 0$

to avoid $\begin{bmatrix} 0 & 0 & \blacksquare \end{bmatrix}$

you must ensure $h-6 \neq 0$

$$h \neq 6$$

$\{\vec{v}_1, \vec{v}_2\}$
 \Leftrightarrow
span \mathbb{R}^2

$$h \neq 6$$

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7. (No Computation) Find 4 vectors that do span \mathbb{R}^3

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

8. (No Computation) Find 4 vectors that do NOT span \mathbb{R}^3

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

9. (No Computation) Can you find 3 vectors that DO span \mathbb{R}^4 ? Why or why not?

No. the columns of
 $[\vec{a}_1, \vec{a}_2, \vec{a}_3]$ span \mathbb{R}^4

\Leftrightarrow
 there is a pivot in each
 row by theorem 4.

But you cannot have 4 pivots in 3 columns.

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Matrices and Matrix Equations

1. If $A = [\vec{a}_1, \dots, \vec{a}_n]$ and $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, write down the formal definition of the matrix product

$$A\vec{x} = [\vec{a}_1, \dots, \vec{a}_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1 \vec{a}_1 + \dots + x_n \vec{a}_n$$

2. Compute the following matrix products. If the product is undefined, explain why.

(a) $\begin{bmatrix} 1 & 2 \\ -2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ DNE. $\begin{array}{l} \text{\# rows of } A \\ \hline \text{columns} \neq \\ \text{\# columns of } \vec{x} \end{array}$

(b) $\begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = (1)\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 2\begin{bmatrix} -2 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} -4 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 9 \end{bmatrix}$
 $= \begin{bmatrix} 0 \\ 13 \end{bmatrix}$

3. Suppose that $A = [\vec{a}_1 \dots \vec{a}_n]$ is an $m \times n$ matrix. Rephrase the sentence "The columns of A span \mathbb{R}^m " as a statement about vectors.

the columns of A span \mathbb{R}^m
 \Leftrightarrow

$x_1 \vec{a}_1 + \dots + x_n \vec{a}_n = \vec{b}$
is consistent
for every \vec{b} in \mathbb{R}^m

4. Suppose that A is an 5×7 matrix, and that $A\vec{x} = \vec{b}$.

Find j and k so that $\vec{x} \in \mathbb{R}^j$ and $\vec{b} \in \mathbb{R}^k$.

$$\begin{array}{c} 5 \times 7 \\ \uparrow \quad \uparrow \\ \text{rows} \quad \text{cols} \\ \text{of } A \quad \text{of } A \end{array} \Rightarrow \left\{ \begin{array}{l} \vec{x} \text{ has 7 rows} \\ \vec{x} \in \mathbb{R}^7 \\ \\ \vec{b} \text{ has 5 rows} \\ \vec{b} \in \mathbb{R}^5 \end{array} \right.$$

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6. Let $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$. Does the system $A\vec{x} = \vec{b}$ has a solution for every $\vec{b} \in \mathbb{R}^2$?

You must justify your answer.

*(full step
Given in
hints)*

is $\left[\begin{array}{cc|c} 2 & 3 & * \\ 1 & 4 & * \end{array} \right]$ always consistent?

$$\sim \left[\begin{array}{cc|c} 1 & 4 & * \\ 2 & 3 & * \end{array} \right]$$

$$\sim \left[\begin{array}{cc|c} 1 & 4 & * \\ 0 & -5 & * \end{array} \right]$$

$[0 \ 0 | \blacksquare]$ cannot occur

\Rightarrow
the system has a solution
for all \vec{b}

7. Let $A = \begin{bmatrix} 3 & 6 & 3 & 0 \\ -3 & -9 & 0 & 3 \\ 0 & 3 & -3 & -3 \end{bmatrix}$.

Determine if the system $A\vec{x} = \vec{b}$ has a solution for every $\vec{b} \in \mathbb{R}^3$.

$$\left[\begin{array}{cccc|c} 3 & 6 & 3 & 0 & * \\ -3 & -9 & 0 & 3 & * \\ 0 & 3 & -3 & -3 & * \end{array} \right]$$

$$\sim \left[\begin{array}{cccc|c} 3 & 6 & 3 & 0 & * \\ 0 & -3 & 3 & 3 & * \\ 0 & 3 & -3 & -3 & * \end{array} \right]$$

$$\sim \left[\begin{array}{cccc|c} 3 & 6 & 3 & 0 & * \\ 0 & -3 & 3 & 3 & * \\ 0 & 0 & 0 & 0 & * \end{array} \right]$$

*No pivot in
row #3* \Rightarrow $[0 \ 0 \ 0 \ 0 | \blacksquare]$
can occur,

so $A\vec{x} = \vec{b}$ does not always have a solution.

NOTE: the system is
neither consistent,
nor inconsistent

because $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ is arbitrary

we see the system is
sometimes but
not always
consistent

1pt

†

Name: _____

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9. (No Computation) Fill in the blanks to state Theorem 4 in terms of pivots.

Theorem 4: The columns of an $m \times n$ matrix A span \mathbb{R}^m

if and only if there is a pivot in every row

10. (No Computation) Write down a 3×3 matrix A whose columns span \mathbb{R}^3 .

$$\begin{bmatrix} 3 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

11. (No Computation) Write down a 3×3 matrix A whose columns do not span \mathbb{R}^3 .

$$\begin{bmatrix} 3 & 0 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

12. (No Computation) Can you write down a 2×3 matrix A whose columns span \mathbb{R}^2 ? Justify your answer.

$$\begin{bmatrix} * & * & * \\ * & * & * \end{bmatrix}_{2 \times 3} \quad \begin{array}{l} \text{columns span } \mathbb{R}^2 \\ \text{has } \xrightarrow{\Leftarrow} \text{pivot in each row} \end{array}$$

Yes. there can be two pivots
because there are two columns

13. (No Computation) Can you write down a 3×2 matrix A whose columns span \mathbb{R}^3 ? Justify your answer.

$$\begin{bmatrix} * & * \\ * & * \\ * & * \end{bmatrix}_{3 \times 2}$$

No. there cannot be 3 pivots
with only ¹⁵ two columns.

Name: _____

Solutions

Section: _____

1. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$

(a) How many solutions are there to the equation $A\vec{x} = \vec{0}$?

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 3 & 4 & 0 \\ 5 & 6 & 0 \end{array} \right] \sim \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & -2 & 0 \\ 0 & -4 & 0 \end{array} \right] \sim \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{array} \right] -$$

there is a unique solution

(b) Let $\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. How many solutions are there to the equation $A\vec{x} = \vec{b}$?

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 3 & 4 & 1 \\ 5 & 6 & 1 \end{array} \right] \sim \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & -2 & -2 \\ 0 & -4 & -4 \end{array} \right] \sim \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & -2 & -2 \\ 0 & 0 & 0 \end{array} \right]$$

there is a unique solution

(c) Let $\vec{c} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. How many solutions are there to the equation $A\vec{x} = \vec{c}$?

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 3 & 4 & 0 \\ 5 & 6 & 1 \end{array} \right] \sim \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & -2 & 0 \\ 0 & -4 & -1 \end{array} \right] \sim \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{array} \right]$$

there are no solutions

(d) How many solutions are *possible* to $A\vec{x} = \vec{d}$, when considering all vectors $\vec{d} \in \mathbb{R}^3$?
You must justify your answer.

there either one or none solutions

/
there is a
pivot in ea.
column

\
there is not
a pivot in
each row.

Name: _____

Section: _____

2. Let $A = \begin{bmatrix} 1 & -2 \\ -3 & 6 \end{bmatrix}$

- (a) How many solutions are there to the equation $A\vec{x} = \vec{0}$?

$$\left[\begin{array}{cc|c} 1 & -2 & 0 \\ -3 & 6 & 0 \end{array} \right] \sim \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

\uparrow no pivot

there are ∞ -many solutions

(b) Let $\vec{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. How many solutions are there to the equation $A\vec{x} = \vec{b}$?

$$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ -3 & 6 & 1 \end{array} \right] \sim \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 0 & 4 \end{array} \right]$$

there are no solutions

(c) Let $\vec{c} = \begin{bmatrix} 5 \\ -15 \end{bmatrix}$. How many solutions are there to the equation $A\vec{x} = \vec{c}$?

$$\left[\begin{array}{cc|c} 1 & -2 & 5 \\ -3 & 6 & -15 \end{array} \right] \sim \left[\begin{array}{cc|c} 1 & -2 & 5 \\ 0 & 0 & 0 \end{array} \right]$$

\uparrow no pivot

there are ∞ -many sol

- (d) How many solutions are *possible* to $A\vec{x} = \vec{d}$, when considering all vectors $\vec{d} \in \mathbb{R}^2$? You must justify your answer.

there are either ∞ -many or no solutions

see column 2
lacks a pivot

row 2 lacks
a pivot

Name: _____

Section: _____

3. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

(a) How many solutions are there to the equation $A\vec{x} = \vec{0}$?

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 0 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -3 & -6 & 0 \end{array} \right]$$

there are ∞ -many solutions \uparrow no pivot

(b) Let $\vec{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. How many solutions are there to the equation $A\vec{x} = \vec{b}$?

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & 1 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -3 & -6 & -3 \end{array} \right]$$

there are ∞ -many solutions \uparrow no pivot

(c) Let $\vec{c} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. How many solutions are there to the equation $A\vec{x} = \vec{c}$?

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & 2 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -3 & -6 & -2 \end{array} \right]$$

~~no pivot~~

there are ∞ -many solutions

(d) How many solutions are possible to $A\vec{x} = \vec{d}$, when considering all vectors $\vec{d} \in \mathbb{R}^2$? You must justify your answer.

ALWAYS

there are ~~no~~ ∞ -many solutions. ~~because~~

Because ~~no~~ pivot in both rows \Rightarrow always has a sln
~~and~~ no pivot in col 3 \Rightarrow has ∞ -many slns.

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10. Write down the formal definition of Linear Dependence of vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly Dependent

If there are c_1, c_2, c_3 in \mathbb{R} not all 0
so that

$$c_1 \cdot \vec{v}_1 + c_2 \cdot \vec{v}_2 + c_3 \cdot \vec{v}_3 = \vec{0}$$

11. Give an example of a non-trivial dependence relation between $\vec{v}_1, \vec{v}_2, \vec{v}_3$.
Use this dependence relation to explain the graphical meaning of "Linear Dependence".

$$2\vec{v}_1 + 6\vec{v}_2 - \vec{v}_3 = \vec{0}$$

solving for $\vec{v}_3 = 2\vec{v}_1 + 6\vec{v}_2$
shows that \vec{v}_3 is in $\text{Span}\{\vec{v}_1, \vec{v}_2\}$.

12. What is the graphical meaning of the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ being Linearly Independent?

If there is no nontrivial dependence relation
Then no \vec{v}_i is in the span of the other ^{remaining} vectors.

Name: _____

Section: _____

4. Determine if the following set of vectors is linearly dependent or independent.

$$\underline{\vec{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}} \quad \vec{a}_2 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \quad \vec{a}_3 = \begin{bmatrix} -2 \\ -2 \\ 2 \end{bmatrix}$$

i.e. is there a nontrivial solution

to $x_1 \vec{a}_1 + x_2 \vec{a}_2 + x_3 \vec{a}_3 = \vec{0}$?

reduce

$$\left[\begin{array}{ccc|c} 1 & 2 & -2 & 0 \\ 1 & 2 & -2 & 0 \\ 1 & 1 & 2 & 0 \end{array} \right]$$

~

$$\left[\begin{array}{ccc|c} 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 4 & 0 \end{array} \right] \begin{matrix} r_2 - r_1 \\ r_3 - r_1 \end{matrix}$$

~

$$\left[\begin{array}{ccc|c} 1 & 2 & -2 & 0 \\ 0 & -1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \begin{matrix} \\ \uparrow \end{matrix}$$

by theorem 2,
3rd column w/o pivot
 \Rightarrow
 ∞ -many slns
 \Rightarrow has nontrivial solution
 $\Rightarrow \boxed{\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}}$ is
 linearly dependent.

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 6 & 0 \\ 0 & -1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\begin{cases} x_1 = -6x_3 \\ x_2 = 4x_3 \\ x_3 \text{ free} \end{cases}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 6 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

Picking $x_3 = 1$ gives one particular solution

$$\Leftrightarrow \begin{cases} x_1 + 6x_3 = 0 \\ x_2 - 4x_3 = 0 \\ x_3 \text{ free} \end{cases}$$

$$\begin{cases} x_1 = -6 \\ x_2 = 4 \\ x_3 = 1 \end{cases}$$

Eg: Nontrivial Dependence Relation

$$(-6) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 4 \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} + 1 \cdot \begin{bmatrix} -2 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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5. Find h so that the vectors $\vec{a}_1 = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$, $\vec{a}_2 = \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}$, and $\vec{a}_3 = \begin{bmatrix} 2 \\ h \\ 1 \end{bmatrix}$ are linearly independent

want ONLY TRIVIAL solution to

$$x_1 \vec{a}_1 + x_2 \vec{a}_2 + x_3 \vec{a}_3 = \vec{0}$$

reduce
$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 3 & 4 & h & 0 \\ 2 & 5 & 1 & 0 \end{array} \right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & h-6 & 0 \\ 0 & 3 & -3 & 0 \end{array} \right] \begin{matrix} r_2 - 3r_1 \\ r_3 - 2r_1 \end{matrix}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & h-6 & 0 \\ 0 & 1 & -1 & 0 \end{array} \right] \begin{matrix} \\ r_3 - r_2 \end{matrix}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & h-6 & 0 \\ 0 & 0 & \underline{-1-(h-6)} & 0 \end{array} \right] \begin{matrix} \\ \\ \uparrow \end{matrix} \begin{matrix} \\ r_3 - r_2 \end{matrix}$$

NOTICE: has ONLY trivial solution \Leftrightarrow has pivot in column 3

$$\Leftrightarrow -1 - (h-6) \neq 0$$

$$-1 - h + 6 \neq 0$$

$$h \neq 5$$

the set is linearly independent when $h \neq 5$.

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13. Suppose that $x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 = \vec{0}$ has a unique solution.
Is $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ Linearly Independent?

Yes. $x_1 = x_2 = x_3 = 0$ is always a solution

So the only dependence relation
is the trivial one.

So the vectors are linearly independent

14. Suppose that $x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 = \vec{0}$ has infinitely many solutions.
Is $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ Linearly Independent?

No. ∞ -many solutions
 \Rightarrow has some nontrivial solution
 \Rightarrow there is a nontrivial dependence relation

15. Does $x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 = \vec{0}$ always have a solution? Why or why not?

$$\begin{aligned} \text{Yes. } 0 \cdot \vec{v}_1 + 0 \cdot \vec{v}_2 + 0 \cdot \vec{v}_3 \\ = \vec{0} + \vec{0} + \vec{0} \\ = \vec{0} \end{aligned}$$

So $x_1 = x_2 = x_3 = 0$ is always a solution.

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11. For each of the following sets of vectors (1) determine if they are linearly independent

You must justify each answer.

$$(a) U = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

consider $\begin{bmatrix} 1 & 1 & 1 & | & 0 \\ 0 & 1 & 1 & | & 0 \\ 0 & 0 & 1 & | & 0 \end{bmatrix} \quad A\vec{x} = \vec{0}$

pivot in each column \Rightarrow has unique soln
 \Rightarrow has only trivial solution
 \Rightarrow columns are INDEPENDENT

$$(b) U = \left\{ \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 6 \end{bmatrix} \right\}$$

shortcut

$$\vec{v}_2 = \frac{1}{2}\vec{v}_1.$$

\Rightarrow is linearly
DEPENDENT

OR

Solve $A\vec{x} = \vec{0}$

$$\begin{bmatrix} 2 & 1 & 0 & | & 0 \\ 0 & 0 & -2 & | & 0 \\ 4 & 2 & 6 & | & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & 0 & | & 0 \\ 0 & 0 & -2 & | & 0 \\ 0 & 0 & 6 & | & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & 0 & | & 0 \\ 0 & 0 & -2 & | & 0 \\ 0 & 0 & 1 & | & 0 \end{bmatrix}$$

$\sim \begin{bmatrix} 2 & 1 & 0 & | & 0 \\ 0 & 0 & -2 & | & 0 \\ 0 & 0 & 1 & | & 0 \end{bmatrix}$
 lacks pivot \Rightarrow has ∞ many solns
 \Rightarrow has some nontrivial soln.

$$(c) U = \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 6 \end{bmatrix} \right\}$$

must row reduce:

$$\begin{bmatrix} 1 & 4 & 3 & | & 0 \\ 0 & 3 & 2 & | & 0 \\ 2 & 5 & 6 & | & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 3 & | & 0 \\ 0 & 3 & 2 & | & 0 \\ 0 & -3 & 0 & | & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 3 & | & 0 \\ 0 & 3 & 2 & | & 0 \\ 0 & 0 & 2 & | & 0 \end{bmatrix}$$

pivot in each
column
 \Rightarrow has only trivial
solution
 \Rightarrow is INDEPENDENT

$$(d) U = \left\{ \begin{bmatrix} 2 \\ 6 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix} \right\}$$

shortcut:

$n=2$ & neither is multiple of
the other

\Rightarrow is INDEPENDENT

OR

Solve $A\vec{x} = \vec{0}$

$$\begin{bmatrix} 2 & 0 & | & 0 \\ 6 & 4 & | & 0 \\ 0 & 4 & | & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & | & 0 \\ 0 & 4 & | & 0 \\ 0 & 4 & | & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & | & 0 \\ 0 & 1 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix}$$

pivot in each coeff column
 \Rightarrow has only trivial solution

\Rightarrow is INDEPENDENT

$$(e) U = \left\{ \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} \right\}$$

Solve $A\vec{x} = \vec{0}$

$$\begin{bmatrix} -1 & 2 & 1 & | & 0 \\ 0 & 4 & 1 & | & 0 \\ 0 & 0 & 1 & | & 0 \end{bmatrix}$$

is in echelon
form already

no pivot in 4th column
 \Rightarrow has ∞ -many solns
 \Rightarrow is DEPENDENT

more columns than rows

⑦

is DEPENDENT